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Abstract: This study focused on linear convection-diffusion problems with constant coefficients with a view of providing the solution with the help of Mahgoub Transform. Convection-diffusion problems can be used to model the behavior of flow phenomena such as momentum and heat, and as well describe the diffusion process in environmental science, such as pollutant transport in the atmosphere, oceans, lakes, rivers or groundwater. Two illustrative examples were considered and Mahgoub Transform method was used to obtain the approximate solutions of the respective examples. The results obtained by the method were compared with the already existing exact solutions and were found to be reliable as the comparison shows a perfect agreement. It was therefore recommended that Mahgoub Transform method can be applied to solve any other linear convection-diffusion problems with constant coefficients.

Keywords: Convection-diffusion equations, linear, Mahgoub Transform, partial derivatives

Introduction

Convection-diffusion equation is the merging of the diffusion and convection equations which describes the physical systems whereby particles, energy, mass, time, length, amount of substance, velocity, density and other physical quantities are transported into the physical field via diffusion and convection. The general form of convection-diffusion equation is given as,

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) - \nabla \cdot (\vec{v} u) + R, \quad (1)$$

Where "u" represents our variable of interest, "D" denotes diffusivity, " \vec{v} " implies average velocity, R is the source of the quantity of "u", " ∇ " means gradient and " $\nabla \cdot$ " means divergence.

Over the years, due to the importance of the convection-diffusion equations to the scientific and non-scientific world, different approaches were adopted to tackle convection-diffusion problems by several researchers, some of which include; Variational Iteration Method (VIM) in (Liu & Zhao, 2010), Adomian Decomposition Method (ADM) in (Momani, An algorithm for solving the fractional convection-diffusion equation with non-linear source term, 2007), Decomposition Method in (Yee, 1993), He's Homotopy Perturbation Method (HHPM) (Momani & Yildirim, Analytical approximate solutions of the fractional convection-diffusion equation with nonlinear source term by He's homotopy perturbation method, 2010), Bessel Collocation Method in (Yuzbasi & Sahin, 2013), Finite-Difference Solutions of Convection-Diffusion Problems in Irregular Domains using a Non-orthogonal Coordinate Transformation in Faghri *et al.* (2007). A Wavelet-Galerkin method for a singularly perturbed convection-dominated diffusion equation in (El-Gamel, 2006), A uniformly convergent scheme on a non-uniform mesh for convection-diffusion parabolic problems in Claveria *et al.* (2003), A uniformly convergent B-spline collocation method on a non-uniform mesh for singularly perturbed one-dimensional time-dependent linear convection-diffusion problem (Kadalbajoo *et al.*, 2008), Laplace Homotopy Perturbation Method in (Gupta *et al.*, 2015). An upwind difference scheme on a novel Shishkin-type mesh for a linear convection diffusion problem (LinB, 1999), Efficient generation of oriented meshes for solving convection-diffusion problems (Madden & Stynes, 1997).

Recently, Mahgoub Transform method of solving ordinary and partial differential equations was introduced by Mohand

Abdelrahim (2016). This method was applied to solve linear ordinary differential equations with constant coefficients in (Mohand, 2016) and was also later employed in solving partial differential equations (Mahgoub & Alshikh, 2017). The definitions, properties and the applications of the Mahgoub Transform to ordinary differential equations are described in (Mohand, 2016) while the definitions, properties and the applications of the Mahgoub transform to linear partial differential equations with constant coefficients are described in (Mahgoub & Alshikh, 2017). The Mahgoub Transform method was therefore applied to solve the convection-diffusion linear problems with constant coefficients in this research work. The Mahgoub Transform method is simple, efficient, and faster compared to previous methods applied in solving convection-diffusion problems. This approach was adopted in this research work due to its less computational work over some other existing approaches.

Materials and Methods

The Mahgoub Transform of any function $f(t)$ can be defined as

$$M\{f(t)\} = H(v),$$

Where:

$$H(v) = v \int_0^\infty f(t) e^{-vt} - vt dt, t \geq 0, \\ k_1 \leq v \leq k_2. \quad (2)$$

Solving for the Mahgoub Transform of partial derivatives, we employed the knowledge of integration by part (Mohand, 2016).

Take,

$$f(x, t) \rightarrow f, H(x, v) = H, M\{f(x, t)\} = H(x, v),$$

$$M\left\{\frac{\partial f(x, t)}{\partial t}\right\} = v \int_0^\infty f(t) e^{-vt} dt,$$

but

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial f}{\partial t} \\ \therefore M\left\{\frac{\partial f}{\partial t}\right\} = v \int_0^\infty \frac{\partial f}{\partial t} e^{-vt} dt = \int_0^\infty v \frac{\partial f}{\partial t} e^{-vt} dt \\ = \lim_{n \rightarrow \infty} \{v e^{-vt} f(x, t)\} \\ + \left\{ \int_0^\infty v e^{-vt} f(x, t) \right\} \\ = v H(x, v) - v f(x, 0). \quad (3)$$

It was assumed that $f(x, t)$ is piecewise continuous and is of exponential order.

Now,

$$M\left\{\frac{\partial f}{\partial t}\right\} = \int_0^\infty v e^{-vt} \frac{\partial f}{\partial t} dt = \frac{\partial f}{\partial x} \int_0^\infty v e^{-vt} f(x, t) dt$$

$$= \frac{\partial f}{\partial x} [H(x, v)] = \frac{d}{dx} [H(x, v)] = \frac{dH}{dx}.$$

Also,

$$M\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right] = M\left[\frac{\partial^2 f}{\partial t^2}\right] = \int v e^{-vt} \frac{\partial^2 f}{\partial t^2} dt,$$

$$\text{let } \frac{\partial f}{\partial t} = T, \quad \therefore \frac{\partial^2 f}{\partial t^2} = \frac{\partial T}{\partial t}.$$

$$M\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right] = \lim_{n \rightarrow \infty} \left\{ \left[v e^{-vt} \frac{\partial f}{\partial t} \right]_0^n + \int v e^{-vt} \frac{\partial f}{\partial t} dt \right\}$$

$$= -v f'(x, 0) + v [v H(x, v) - v f(x, 0)]$$

$$= v^2 H(x, v) - v^2 f(x, 0) - v f'(x, 0). \quad (4)$$

This result may be extended to the n^{th} order partial derivatives by using mathematical induction. Now, considering the general form of the convection-diffusion equation in (Gupta, Kumar, & Singh, 2015), and applying the Mahgoub Transform in (Mohand, 2016), we have

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) - \nabla \cdot (\vec{v} \cdot u) + R. \quad (5)$$

Applying the Mahgoub Transform to both sides of equation (5), we obtain

$$M\left[\frac{\partial u}{\partial t}\right] = M[\nabla \cdot (D \nabla u)] - M[\nabla \cdot (\vec{v} \cdot u)] + M[R], \quad (6)$$

Then, we take the inverse Mahgoub Transform of both sides in equation (6) to obtain our expected result.

Results and Discussion

This section demonstrates the application of the Mahgoub Transform on the linear convection-diffusion equation. The authenticity of the Mahgoub Transform method was examined by comparing the solution obtained with the reviewed solutions in the following two applications.

Application 1

Solve the diffusion-convection problem (Gupta *et al.*, 2015)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \quad (7)$$

with the initial condition

$$u(x, 0) = x + e^{-x} \quad (8)$$

Applying the Mahgoub Transform to both sides of equation (7) gives

$$M\left[\frac{\partial u}{\partial t}\right] = M\left[\frac{\partial^2 u}{\partial x^2} - u\right]. \quad (9)$$

Observing the rules of linearity on equation (9) yields

$$M\left[\frac{\partial u}{\partial t}\right] = M\left[\frac{\partial^2 u}{\partial x^2}\right] - M[u],$$

$$vH(x, v) - vu(x, 0) = \frac{d^2 H(x, v)}{dx^2} - H(x, v). \quad (10)$$

Applying the initial condition from equation (8) into (10) produces

$$vH(x, v) - v(x + e^{-x}) = \frac{d^2 H(x, v)}{dx^2} - H(x, v)$$

$$\frac{d^2 H(x, v)}{dx^2} + [-1 - v]H(x, v) = -v(x + e^{-x}). \quad (11)$$

Now, it is so clear that equation (11) gives an expression of non-homogeneous second order ordinary differential equation which may be solved by the method of undetermined coefficient.

Assume the general solution of (11) to be

$$H(x, v) = H_c(x, v) + H_p(x, v), \quad (12)$$

Where $H_c(x, v)$ is the complementary function and $H_p(x, v)$ is the particular integral. Taking the Left-Hand Side of

equation (11) as equal to zero, then the auxiliary equation becomes

$$m^2 + [-1 - v] = 0, \quad m^2 = 1 + v,$$

$$m = \pm\sqrt{1+v},$$

$$m_1 = +\sqrt{1+v}, \quad m_2 = -\sqrt{1+v},$$

then the complementary function to the auxiliary equation is given as

$$H_c(x, v) = c_1 e^{m_1 x} + c_2 e^{m_2 x}, \quad (13)$$

Where: c_1 and c_2 are arbitrary constants and m_1 and m_2 are the roots of the auxiliary equation.

Substituting m_1 and m_2 into equation (13) gives

$$H_c(x, v) = c_1 e^{(\sqrt{1+v})x} + c_2 e^{-(\sqrt{1+v})x}. \quad (14)$$

Considering the right-hand side of equation (11), i.e. solving for the non-homogeneous part, we assign our particular integral as

$$H_p(x, v) = Ax + B + De^{-x}, \quad (15)$$

$$H'_p(x, v) = A - De^{-x}, \quad (16)$$

$$H''_p(x, v) = De^{-x}, \quad (17)$$

where A, B, D are all arbitrary constants. Upon substituting equations (15), (16), and (17) all in equation (11) and solving using comparison of co-efficient to obtain

$$De^{-x} + (-1 - v)[Ax + B + De^{-x}] = -vx - ve^{-x}.$$

Expand the brackets from the L.H.S to have

$$De^{-x} - Ax - B - De^{-x} - Avx - Bv - vDe^{-x}$$

$$= -vx - ve^{-x}.$$

Upon simplification,

$$-Ax - B - Avx - Bv - vDe^{-x} = -vx - ve^{-x}.$$

Equating the co-efficient of

$$e^{-x}: -Dv = -v \Rightarrow D = 1,$$

$$x: -A - Av = -v \Rightarrow A[-1 - v] = -v,$$

$$A = \frac{v}{1+v},$$

constant: $-B - Bv = 0, B(-1 - v) = 0, B = 0.$

Therefore, the particular integral $H_p(x, v)$ is given as

$$H_p(x, v) = \frac{vx}{1+v} + 0 + 1 \cdot e^{-x} = \frac{vx}{1+v} + e^{-x}. \quad (18)$$

Now let's substitute equations (14) and (18) in equation (12) to have

$$H(x, v) = c_1 e^{(\sqrt{1+v})x} + c_2 e^{-(\sqrt{1+v})x} + \frac{vx}{1+v} + e^{-x}. \quad (19)$$

Since $H(x, v)$ is bounded, then $c_1 = c_2 = 0.$

$$\therefore H(x, v) = \frac{vx}{1+v} + e^{-x}. \quad (20)$$

Taking the inverse Mahgoub Transform of equation (20) to obtain

$$M^{-1}[H(x, v)] = M^{-1}\left[\frac{vx}{1+v}\right] + M^{-1}[e^{-x}].$$

$$\therefore U(x, t) = xe^{-t} + e^{-x}.$$

This result corresponds with the exact solution

$$U(x, t) = xe^{-t} + e^{-x},$$

as obtained in Gupta *et al.* (2015).

Application 2

Consider the following diffusion-convection problem (Gupta *et al.*, 2015)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 14u, \quad x, t \in \mathbb{R}, \quad (21)$$

With the initial condition

$$u(x, 0) = \frac{1}{2}x + e^{-\frac{x}{2}}. \quad (22)$$

The exact solution of this problem was given as $u(x, t) = e^{-\frac{x}{2}} + \frac{x}{2}e^{-\frac{t}{4}}$ in (Gupta *et al.*, 2015).

Take the Mahgoub Transform of both sides of equation (21) to get

$$M\left[\frac{\partial u}{\partial t}\right] = M\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}u\right]. \quad (23)$$

Applying linearity condition on equation (23), we have

$$M\left[\frac{\partial U}{\partial t}\right] = M\left[\frac{\partial^2 U}{\partial x^2}\right] - M\left[\frac{1}{4}U\right] = M\left[\frac{\partial^2 U}{\partial x^2}\right] - \frac{1}{4}M[U],$$

$$vH(x, v) - vU(x, 0) = \frac{d^2 H(x, v)}{dx^2} - \frac{1}{4}H(x, v)$$

$$\frac{d^2 H(x, v)}{dx^2} - \frac{1}{4}H(x, v) - vH(x, v) = -vU(x, 0). \quad (24)$$

Applying the initial condition in equation (22) one equation (24) yields

$$\frac{d^2 H(x, v)}{dx^2} - \frac{1}{4}H(x, v) - vH(x, v) = -vU(x, 0)$$

$$= -v\left[\frac{1}{2}x + e^{-\frac{x}{2}}\right],$$

$$\frac{d^2 H(x, v)}{dx^2} + H(x, v)\left[-\frac{1}{4} - v\right]$$

$$= -v\left[\frac{1}{2}x + e^{-\frac{x}{2}}\right]. \quad (25)$$

Now, it is so clear that equation (25) gives an expression of non-homogeneous second order ordinary differential equation which may be solved by the method of undetermined coefficient.

Assume the general solution of equation (25) to be

$$H(x, v) = H_c(x, v) + H_p(x, v), \quad (26)$$

Where $H_c(x, v)$ is the complementary function and $H_p(x, v)$ is the particular integral. Equating the Left-Hand Side of equation (25) to zero, then the auxiliary equation becomes

$$m^2 + \left[-\frac{1}{4} - v\right] = 0, \quad m^2 = \frac{1}{4} + v,$$

$$m = \pm \sqrt{\frac{1}{4} + v}, \quad m_1 = \sqrt{\frac{1}{4} + v}, m_2$$

$$= -\sqrt{\frac{1}{4} + v},$$

Then, the complementary function to the auxiliary equation is given as

$$H_c(x, v) = c_1 e^{m_1 x} + c_2 e^{m_2 x}, \quad (27)$$

Where: c_1 and c_2 are arbitrary constants and m_1 and m_2 the roots of the auxiliary equation. Substituting m_1 and m_2 into equation (27) gives

$$H_c(x, v) = c_1 e^{\left(\sqrt{\frac{1}{4} + v}\right)x} + c_2 e^{-\left(\sqrt{\frac{1}{4} + v}\right)x}. \quad (28)$$

Considering the Right-Hand Side of equation (25), i.e. solving for the non-homogeneous part, we assign our particular integral as

$$H_p(x, v) = Ax + B + De^{-\frac{x}{2}}, \quad (29)$$

$$H'_p(x, v) = A - \frac{1}{2}De^{-\frac{x}{2}}, \quad (30)$$

$$H''_p(x, v) = \frac{1}{4}De^{-\frac{x}{2}}, \quad (31)$$

Where: A, B, D are all arbitrary constants.

Substituting equations (29), (30) and (31) in equation (25) and solving using comparison of co-efficient method gives

$$\frac{1}{4}De^{-\frac{x}{2}} + \left[-\frac{1}{4} - v\right]\left[Ax + B + De^{-\frac{x}{2}}\right]$$

$$= -v\left[\frac{1}{2}x + e^{-\frac{x}{2}}\right].$$

Expanding the brackets from the L.H.S yields

$$\frac{1}{4}De^{-\frac{x}{2}} - \frac{Ax}{4} - \frac{B}{4} - \frac{De^{-\frac{x}{2}}}{4} - Avx - Bv - Dve^{-\frac{x}{2}}$$

$$= -v\frac{1}{2}x - ve^{-\frac{x}{2}}.$$

Upon simplification,

$$-\frac{Ax}{4} - \frac{B}{4} - Avx - Bv - Dve^{-\frac{x}{2}} = -v\frac{1}{2}x - ve^{-\frac{x}{2}}.$$

Equating the co-efficient of

$$e^{-x/2}: -Dv = -v \Rightarrow D = 1,$$

$$x: -\frac{A}{4} - Av = -\frac{v}{2} \Rightarrow A\left[-\frac{1}{4} - v\right] = -\frac{v}{2},$$

$$A = \frac{2v}{4v + 1},$$

$$\text{constant: } -\frac{B}{4} - Bv = 0, \quad B\left(-\frac{1}{4} - v\right) = 0, \quad B = 0.$$

Therefore, the particular integral $H_p(x, v)$ is given as

$$H_p(x, v) = \frac{2vx}{4v + 1} + 0 + 1 \cdot e^{-\frac{x}{2}} = \frac{2vx}{4v + 1} + e^{-\frac{x}{2}} \quad (32)$$

Now let's substitute equations (28) and (32) into equation (26) to obtain

$$H(x, v) = c_1 e^{\left(\sqrt{\frac{1}{4} + v}\right)x} + c_2 e^{-\left(\sqrt{\frac{1}{4} + v}\right)x} + \frac{2vx}{4v + 1} + e^{-\frac{x}{2}}.$$

Since $H(x, v)$ is bounded, then $c_1 = c_2 = 0$.

$$\therefore H(x, v) = \frac{2vx}{4v + 1} + e^{-\frac{x}{2}}. \quad (33)$$

Taking the inverse Mahgoub Transform of equation (33) yields

$$M^{-1}[H(x, v)] = M^{-1}\left[\frac{2vx}{4v + 1}\right] + M^{-1}\left[e^{-\frac{x}{2}}\right].$$

$$\therefore U(x, t) = \frac{x}{2}e^{-\frac{t}{4}} + e^{-\frac{x}{2}}.$$

This result is the same with the exact solution

$$U(x, t) = \frac{x}{2}e^{-\frac{t}{4}} + e^{-\frac{x}{2}},$$

gotten in Gupta *et al.* (2015).

Conclusion

In this paper, Mahgoub Transform method was used to obtain the solutions of two different linear convection-diffusion problems with constant co-efficient. This method yields a result which conforms to the exact solutions of the illustrative examples considered in this work. Hence, we therefore showed that this method is applicable to every other problems of linear convection-diffusion equations with constant co-efficient.

Conflict of Interest

Authors have declared that there is no conflict of interest reported in this work.

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